RESIDUAL LIFELENGTHS OF CENSORED DATA

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RESIDUAL LIFELENGTHS OF CENSORED DATA

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We have read the thesis entitled **“RESIDUAL LIFELENGTHS OF CENSORED DATA ”** completed by **CEKİ FRANKO** under supervision of **Prof. Dr. İsmihan Bayramoğlu** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

RESIDUAL LIFELENGTHS OF CENSORED DATA

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Residual lifelengths of the remaining components are of special in reliability analysis, actuarial science and survival studies. In this thesis the residual lifetime of remaining components under different types of censoring schemes are investigated. Some interesting results about the distributional properties, mean residual lives and characterizations are obtained.

Keywords: Order Statistics; Residual Lifetimes; Mean Residual life; $(n-k+1)$-out-of-n system; Sequential $(n-k+1)$-out-of-n system.

ÖZ

SANSÜRLENMİŞ VERİLERİN GERİYE KALAN YAŞAM UZUNLUKLARI

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Geriye kalan elemanların kalan yaşam uzunlukları güvenilirlik analizinde, aktüerya biliminde ve sağ kalan analizinde büyük önem teşkil etmektedir. Bu tezde farklı sansürleme şemaları altında geriye kalan elemanların kalan yaşam uzunlukları araştırıldı. Dağılım özellikleri, ortalama geriye kalan yaşamlar ve karakterizasyonlarla ilgili bazı ilginç sonuçlar elde edildi.

Anahtar Kelimeler: Sıra istatistikleri; Geriye kalan yaşam süreleri; Ortalama Geriye kalan yaşam; n'in $(n-k+1)$'li sistem; Dizili n'in $(n-k+1)$'li sistem.

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To my family

Contents

Chapter 1

Introduction

The subject of order statistics is one of the most important concepts of the statistical theory. It is used in both theoretical and application areas. Order statistics are sufficient statistics, hence they contain all the information about the sample. Moreover since most statistics based on order statistics, have the distribution-free property, it is widely used in non-parametric statistical methods. Another important aspect of the order statistics is that, the lifetime of the components can be represented with order statistics so it takes a major place in lifetime analysis and reliability theory. In lifetime analysis and reliability experiments there are different scenarios involving the components that are tested removed from the experiment before failure. Such components are generally called censored components. The reason behind these scenarios is to rescue the unfailed components for possible future use in other systems. Another reason is that the tested components may be expensive so it is reasonable to stop the experiment before waiting all of them to fail and have an insight about the lifetimes of the rescued ones. In life-time analysis and reliability theory two censoring schemes, Type-I and Type-II are studied with intensity. Assume that at time 0 n components are put on a life-time test. Type-I censoring occurs when a pre-fixed time t is reached. At time t the experiment is stopped. It should be noted that t is independent of the failure times. In Type-I censoring the failure of a component can be observed if it fails before t. The survived components, in other words components which have a life-time greater than t, are called censored units. In Type-II censoring, the experiment is terminated when a prefixed number say, the k^{th} failure occurs. In this scheme the remaining $n-k$ components are called censored components.

In reliability theory an $(n-k+1)-$out-of-n system functions until k components have failed so it is reasonable to test them in Type-II censoring scheme and investigate the lifetimes of the remaining $n-k$ censored components. In the classical theory of $(n-k+1)-$out-of-n systems it is assumed that the lifetimes of the components are independent and identically distributed and the failure of one component does not affect the functioning of the remaining ones. If this is not the case, for instance the failure of a component puts added stress or load on the remaining components, then models involving sequential order statistics (Kamps, 1995) can be used in the analysis of such systems. Another scenario occurs when additional components are removed after the failure of a component. If this is the case, progressively type-II right censored order statistics can be applied effectively to reflect the situation. In this paper the marginal and joint distributions of residual lifetimes of the remaining components are obtained for different types of censored data and some distributional properties and characterizations are investigated.

Chapter 2

Order Statistics

The independent and identically distributed random variables which can be interpreted as results of an experiment measuring values of a certain random variable arranged in order of magnitude, are called order statistics. In the statistical model of many experiments, for instance in reliability analysis, life time studies, the analysis of time to graduation of students and testing of strength of materials, the realizations arise in nondecreasing order, therefore the use of order statistics is necessary. Order statistics are extensively used in statistical inferences; in estimation theory and hypothesis testing. Order statistics have wide applications in many areas where the use of an ordered sample is important. Let $X_1, X_2, ..., X_n$ denote a random sample from a population with cumulative distribution function (c.d.f) F Suppose that the elements of this sample are arranged in order of magnitude and $X_{(1)}$ denotes the smallest; $X_{(2)}$ denotes the second smallest; etc. and $X_{(n)}$ denotes the largest of the set $X_1, X_2, ..., X_n$. Then $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denotes the original random sample arranged in increasing order of magnitude, and these are called the order statistics associated with the sample $X_1, X_2, ..., X_n$. We call $X_{(i)}$, for $1 \leq i \leq n$ the i^{th} order statistic. The subject of order statistics deals with the distributional properties of $X_{(i)}$ itself, some functions of the subset of the n order statistics and their applications. It is well known from classical statistical theory, that the natural estimate of an unknown distribution function is the empirical distribution function, which is a function of order statistics. Therefore, many important statistics in estimation theory and hypothesis testing appear to be an integral functional of the empirical distribution function, and can be expressed in terms of order statistics. Order statistics do not change their order under probability integral transformation, namely if $U_{(i)} = F(X_{(i)}),\ \ i = 1, 2, ..., n$ then $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$. Due to unique distribution free properties, they are widely used in nonparametric interval estimation and hypothesis testing. Order statistics and their properties have been extensively studied since early part of the last century, and recent years have seen a particularly rapid growth of studies. The multiauthored book *Contributions to Order Statistics,* edited by A.H. Sarhan and B.G. Greenberg appeared in the Willey series in probability and statistics in 1962. The first monograph, *"Order Statistics"* by H. David appeared in 1970 in the same Willey series and has served as a text, a survey of growth and a general introduction. The Second edition appeared in 1981 and the third, coauthored with H. Nagaraja, in 2003. For further reading we refer also Arnold et al. (1992), Balakrishnan (2007).

2.1 Distribution Theory of Order Statistics

Let $X_1, X_2, ..., X_n$ be a sample size of n from the population with c.d.f F (i.i.d. random variables with c.d.f F).
The order statistics obtained by arranging the random sample $X_1, X_2, ..., X_n$ in increasing order of magnitude are denoted by

$$X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$$

or

$$X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}.$$

The distribution function of the r^{th} order statistic is

$$F_{r:n}(x) = P\left\{X_{r:n} \leq x\right\} = \sum_{i=r}^{n} \binom{n}{i} F^i(x) \left(1 - F(x)\right)^{n-i}. \tag{2.1}$$

If F is absolutely continuous with pdf f then (2.1) can be written also as follows

$$F_{r:n}(x) = \frac{1}{B(r,n-r+1)} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du$$
$$= \frac{1}{B(r,n-r+1)} I_{F(x)}(r, n-r+1), \qquad (2.2)$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt,$$
$$I_p(a,b) = \int_0^p t^{a-1}(1-t)^{b-1} dt,$$

and the coefficient $\frac{1}{B(r,n-r+1)} = \frac{n!}{(r-1)!(n-r)!}$.

Formula (2.1) yields true for discrete, absolutely continuous and continuous except countable number of points (having countable number points of discontinuity). Formula (2.2) is true only for absolutely continuous distribution. Given the realizations of the n order statistics to be $X_{1:n} < X_{2:n} < ... < X_{n:n}$, the original random variables X_i are restrained to take on the values $X_{i:n}$ $(i = 1, 2, .., n)$ which by symmetry assigns equal probability to each of the $n!$ permutations of (1,2,...,n). Therefore, the joint density function of all n order statistics is

$$f_{1,2,...,n}(x_1, x_2, ..., x_n) = \begin{cases} n! \prod_{i=1}^{n} f(x_i) & \text{if } x_1 < x_2 < ... < x_n \\ 0 & \text{otherwise} \end{cases} \qquad (2.3)$$

The joint pdf of two or more order statistics can be obtained by integrating from (2.3) as well as by using continuous total probability formula.

The joint pdf of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is

$$f_{r,s}(x,y) = \begin{cases} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F^{r-1}(x) \\ \times (F(y) - F(x))^{s-r-1}(1-F(y))^{n-s} f(x) f(y) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases} \qquad (2.4)$$

The joint pdf of order statistics $X_{r_1:n}, X_{r_2:n}, ..., X_{r_k:n}$ is

$$f_{r_1,r_2,...,r_k}(x_1, x_2, ..., x_k)$$

$$= \begin{cases} \frac{n!}{(r_1-1)!(r_2-r_1-1)!...(n-r_k)!} F^{r_1-1}(x_1)\left[F(x_2)-F(x_1)\right]^{r_2-r_{1-1}} \\ \quad \times \left[F(x_3)-F(x_2)\right]^{r_3-r_2-1} ... \left[1-F(x_k)\right]^{n-r_k} \\ \quad\quad \times f(x_1)...f(x_k) & \text{if } x_1 < x_2 < ... < x_k \\ \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

If F is a discrete distribution function, then the joint c.d.f of $X_{r:n}$ and $X_{s:n}$ is

$$F_{r,s}(x,y) = \begin{cases} \sum_{i=r}^{n} \sum_{j=\max(0,s-i)}^{n-i} \frac{n!}{i!j!(n-i-j)!}(F(x))^i \\ \quad \times [F(y)-F(x)]^j [1-F(y)]^{n-i-j} & \text{if } x < y \\ \\ F_s(y) & \text{otherwise} \end{cases} \tag{2.6}$$

and the pmf of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s}(x,y) = F_{r,s}(x,y) - F_{r,s}(x-1,y) - F_{r,s}(x,y-1) + F_{r,s}(x-1,y-1), \ \ x \leq y. \tag{2.7}$$

Definition. Let $X_{1:n}, ..., X_{n:n}$ be order statistics based on the sample X_1, $X_2, ...,$ X_n. Then the random variables

$$Y_1 = X_{1:n}, Y_2 = X_{2:n} - X_{1:n}, \ , ..., Y_n = \ X_{n:n} - X_{n-1:n}$$

are called spacings.

Let $X_{1:n}, ..., X_{n:n}$ be order statistics based on the sample X_1, $X_2, ...,$ X_n with c.d.f $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$. Then the spacings

$$Y_1 = X_{1:n}, Y_2 = X_{2:n} - X_{1:n}, \ , ..., Y_n = \ X_{n:n} - X_{n-1:n}$$

are independent, furthermore the random variables

$$Z_1 = n\lambda X_{1:n}, Z_2 = (n-1)\lambda(X_{2:n} - X_{1:n}) \ , ...,$$
$$Z_r = \ (n-r+1)\lambda(X_{r:n} - X_{r-1:n}), ..., Z_n = \ \lambda(X_{n:n} - X_{n-1:n})$$

are i.i.d. with $F(x) = 1 - \exp(-x)$, $x \geq 0$.
If n units placed under test of solidity and $X_1, X_2, ..., X_n$ represent the life lengths of these units, then the lengths of time intervals $X_{r:n} - X_{r-1:n}$,$r = 1, 2, ..., n$ between two failures are independent and identically distributed random variables.

Theorem 2.1 *Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be order statistics of the sample $X_1, X_2, ..., X_n$ with absolutely continuous c.d.f F and p.d.f f. Then*

$$\{(X_{r+1:n}, X_{r+2:n}, ..., X_{n:n}) \mid X_{r:n} = x\} \overset{d}{=} (Y_{1:n-r}, Y_{2:n-r}, ..., Y_{n-r:n-r}),$$

where $Y_{1:n-r}, Y_{2:n-r}, ..., Y_{n-r:n-r}$ are order statistics from the sample $Y_1, Y_2, ..., Y_{n-r}$ of size $n - r$, and

$$Y \overset{d}{=} \{X \mid X > x\}$$

$$\text{the pdf of } Y \text{ is } f_Y(u) = \begin{cases} 0 & \text{if } u \leq x \\ \frac{f(u)}{1-F(x)} & \text{otherwise} \end{cases}.$$

Theorem 2.2 *Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be order statistics of the sample $X_1, X_2, ..., X_n$ with absolutely continuous c.d.f F and p.d.f f. Then*

$$\{(X_{r+1:n}, X_{r+2:n}, ..., X_{s-1:n}) \mid X_{r:n} = x, X_{s:n} = y\}$$
$$\overset{d}{=} (Y_{1:s-r-1}, Y_{2:s-r-1}, ..., Y_{s-r-1:s-r-1}),$$

where $Y_{1:s-r-1}, Y_{2:s-r-1}, ..., Y_{s-r-1:s-r-1}$ are order statistics from the sample $Y_1, Y_2, ..., Y_{s-r}$ size $s - r - 1$, and

$$Y \overset{d}{=} \{X \mid x \leq X \leq y\}$$

$$\text{the pdf of } Y \text{ is } f_Y(u) = \begin{cases} 0 & \text{if } u \notin [x, y] \\ \frac{f(u)}{F(y)-F(x)} & \text{otherwise} \end{cases}.$$

Theorem 2.3 *Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be order statistics of the sample $X_1, X_2, ..., X_n$ with absolutely continuous c.d.f F and p.d.f f. Then*

$$\{X_{j:n} \mid X_{j-p:n} = x, X_{j+q:n} = y\} \overset{d}{=} Y_{p:p+q-1}$$
$$p + 1 \leq j \leq n - q$$

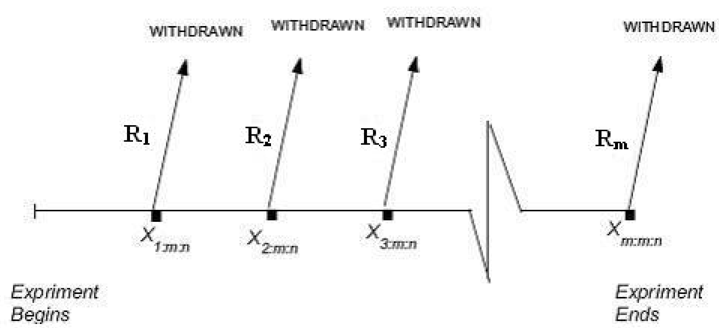
where

$$Y \overset{d}{=} \{X \mid x \leq X \leq y\}.$$

(Bairamov and Özkal (2007)).

2.2 Progressive Type II censored order statistics

The model of Progressive Type II censored order statistics is one of the most applicable general models of ordered random variables and is very useful in reliability and life time studies. Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables representing failure times of n identical units placed on a life test. Under the Progressive Type II right censoring scheme, at the time of i^{th} failure R_i $(i = 1, 2, ..., m$ and $m \leq n)$ surviving items are removed at random from the experiment, where $m + \sum_{i=1}^{n} R_i = n$. Let $\mathbf{R} = (R_1, R_2, ..., R_m)$. Denote the m ordered observed failure times by $X_{1:m:n}$, $X_{2:m:n}$, ..., $X_{m:m:n}$. These random variables are called Progressive Type II right censored order statistics from a sample $X_1, X_2, ..., X_n$ with progressive censoring scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$. A nice description of details of theory, methods and applications of Progressive censoring can be found in Balakrishnan and Aggarwala (2000). Schematic representation of Progressive Type-II right censoring experiment with censoring scheme $R = (R_1, R_2, ..., R_m)$:



If the failure times of the n items originally on the test are from a continuous population with c.d.f F and p.d.f f, the joint p.d.f of all m progressively Type II censored order statistics is

$$f_{1,2,...,m}(x_1, x_2, ..., x_m) = c \sum_{i=1}^{m} f(x_i) \{1 - F(x_i)\}^{R_i}, \; x_1 < x_2 < ... < x_m,$$

where $c = n(n - R_1 - 1)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$.

The marginal c.d.f and p.d.f of $X_{r:m:n}$ are given respectively as

$$F_{X_{r:m:n}}(x) = 1 - c_{r-1}\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}(1-F(x))^{\gamma_i}\ ,\ 1 \leq r \leq m$$

and

$$f_{X_{r:m:n}}(x) = c_{r-1}f(x)\sum_{i=1}^{r}a_{i,r}(1-F(x))^{\gamma_i}\ ,\ 1 \leq r \leq m$$

where $c_{r-1} = \prod_{j=1}^{m}\gamma_j$ and $a_{i,r} = \prod_{\substack{j=1 \\ j\neq i}}^{r}\frac{1}{\gamma_j-\gamma_i}, 1 \leq i \leq r \leq m, m \geq 2,\ \gamma_j = n-j+\sum_{i=1}^{n}R_i+1$ and the empty product Π_ϕ is defined to be 1.

2.3 Sequential Order Statistics

Definition. (Kamps 1995) Let $\left(Y_j^{(i)}\right)$, $1 \leq j \leq n-i+1, 1 \leq i \leq n$, be a sequence of independent random variables with $\left(Y_j^{(i)}\right) \sim F_i, 1 \leq j \leq n-i+1, 1 \leq i \leq n$, where $F_1, \ldots F_n$ are continuos distribution functions with $F_1^{-1}(1) \leq \ldots \leq F_n^{-1}(1)$. Let

$$\begin{aligned} X_j^{(1)} &= Y_j^{(1)}, \quad 1 \leq j \leq n, \\ X_*^{(1)} &= \min\{X_1^{(1)}, \ldots, X_n^{(1)}\}, \end{aligned}$$

and for $2 \leq i \leq n$, let

$$\begin{aligned} X_j^{(i)} &= F_i^{-1}\left(F_i(Y_j^{(i)})(1-F_i(X_*^{(i-1)})) + F_i(X_*^{(i-1)})\right), 1 \leq j \leq n-i+1, \\ X_*^{(i)} &= \min\{X_1^{(i)}, \ldots, X_{n-i+1}^{(i)}\}. \end{aligned}$$

Then the random variables $X_*^{(1)}, X_*^{(2)}, \ldots, X_*^{(n)}$ are called sequential order statistics (based on $F_1, F_2, \ldots, F_n$).
One can obtain the conditional distribution of the random variables $X_1^{(i)}$ for

$1 \leq i \leq n$. Since $Y_1^{(i)}$ and $X_*^{(i-1)}$ are independent we have

$$P\left(X_1^{(i)} \leq t | X_*^{(i-1)} = s\right) = P\left(F_i(Y_1^{(i)}) \leq \frac{F_i(t) - F_i(s)}{1 - F_i(s)}\right)$$
$$= \frac{F_i(t) - F_i(s)}{1 - F_i(s)}, \text{ for } t \geq s.$$

For a particular choice of the distribution functions $F_1, \ldots, F_n$, namely

$$F_i(t) = 1 - (1 - F(t))^{\alpha_i}, 1 \leq i \leq n,$$

with some absolutely continuous and strictly increasing distribution function F and positive real numbers $\alpha_1, \ldots, \alpha_n$ Kamps(1996) found the joint density of the first r sequential order statistics $X_*^{(1)}, \ldots, X_*^{(r)}$ as

$$f^{X_*^{(1)} \ldots, X_*^{(r)}}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!}\left(\prod_{j=1}^{r} \alpha_j\right) \times \left(\prod_{j=1}^{r-1} (1 - F(x_j))^{m_j} f(x_j)\right)$$
$$\times (1 - F(x_r))^{\alpha_r(n-r+1)-1} f(x_r), \ x_1 < \ldots x_r, r \leq n$$
$$\text{with } m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1} - 1, \ 1 \leq j \leq n-1.$$

Ordinary order statistics are contained in the model of sequential order statistics in the distribution theoretical sense. Choosing $r = n$ and $\alpha_1 = \ldots, = \alpha_n$ we obtain the joint density function of order statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ based on iid random variables $X_1, \ldots X_n$ with distribution function $1 - (1 - F)^{\alpha_1}$.

Chapter 3

Residual Lifetimes of Remaining Components in an $n-k+1$ out of n system

In their article Bairamov and Arnold (2007) found the residual lifelengths of the remaining functioning components following the k^{th} failure in the system. In addition they discuss the joint distribution of these exchangeable random variables and identify the sufficient conditions that guarantee independence of the residual lifelengths.

Consider an $(n-k+1)$ out of n system which will function successfully until k of the components have failed. Consequently, if we denote the lifetimes of the individual components by $X_1, X_2, \ldots, X_n$ then the lifetime of the $(n-k+1)$ out of n system will be represented by the kth order statistic $X_{k:n}$. After an $n-k+1$ out of n system fails (i.e. after the kth failure has been observed), it is often reasonable to break down the system and rescue the functioning components for possible future use in other systems. It will be of interest to determine the joint distribution of the residual lifetimes of these functioning components in order to assess the desirability of reusing them in other systems. In the modelling of failure times for components of the system with i.i.d components, we assume that the failure of one component does not affect the functioning of the remaining ones.

The classical theory of $n-k+1$ out of n systems assumes that the n lifetimes $X_1, X_2, \ldots, X_n$ of the components of the system are independent and identically distributed (i.i.d) with common absolutely continuous distribution function F and corresponding density f. With this setup, the time of the first failure will be the first order statistic $X_{1:n}$ and the subsequent times between failures can be identified with the spacings $X_{i:n} - X_{i-1:n}, i = 2, 3, \ldots, n$. The i.i.d assumption is often crucial for obtaining relatively simple distributional results. Note that even under the classical assumption that the original lifetimes were i.i.d, it will turn out that the residual lifetimes of the unfailed items will be exchangeable, but typically not independent. They will be conditionally independent given the time of the k^{th} failure, but we are not assuming that the time of that failure is known, or equivalently we do not know the time at which the system was switched on, we just know it has stopped functioning because k failures have occurred. Note that if we put the rescued components into a new system, we will need to consider systems with dependent identically distributed component lifetimes, thus justifying a concern with at least this variation on the classical set up of $n-k+1$ out of n systems.

For any $k \in \{1, 2, ..., n\}$ we will use the notation $X_1^{(k)}$, $X_2^{(k)}, ..., X_{n-k}^{(k)}$ to denote the residual lifetimes of the $n-k$ components still functioning at the time of the k^{th} failure. For each k, we may define

$$X_{1:n-k}^{(k)} = \min\{X_1^{(k)}, X_2^{(k)}, \ldots, X_{n-k}^{(k)}\}.$$

Upon reflection, it is evident that these $X_{1:n-k}^{(k)}$'s simply represent an alternative description of the spacings of the order statistics of the original sample $X_1, X_2, \ldots, X_n$. Thus

$$X_{k+1:n} - X_{k:n} = X_{1:n-k}^{(k)}$$

and

$$X_{k-1:n} = X_{1:n} + X_{1:n-1}^{(1)} + X_{1:n-2}^{(2)} + \ldots + X_{1:n-k}^{(k)}.$$

3.1 The joint distribution of the residual lifelengths of remaining components

We begin with $X_1, X_2, \ldots, X_n$ i.i.d with common absolutely continuous distribution F and density f. If we are given $X_{k:n} = x$, then the conditional distribution of the subsequent order statistics $X_{k+1:n}, \ldots, X_{n:n}$ is the same as the distribution of order statistics of a sample of size $n-k$ from the distribution F truncated below at x. If we denote by $Y_i^{(k)}, i = 1, 2, \ldots, n-k$ the randomly ordered values of $X_{k+1:n}, \ldots, X_{n:n}$, then given $X_{k:n} = x$, these $Y_i^{(k)}$'s will be i.i.d with common survival function $\bar{F}(x+y)/\bar{F}(x)$. The residual lifetimes after k failures, $X_1^{(k)}, \ldots, X_{n-k}^{(k)}$, may be represented as

$$X_i^{(k)} = Y_i^{(k)} - X_{k:n}, \quad i = 1, 2, \ldots, n-k$$

and using $F_{k:n}$ to denote the distribution function of $X_{k:n}$ we can obtain the joint survival function of the residual lifelengths as follows

$$\begin{aligned}
\bar{F}_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) &= P(X_1^{(k)} > x_1, X_2^{(k)} > x_2, \ldots, X_{n-k}^{(k)} > x_{n-k}) \\
&= \int_0^\infty P(X_1^{(k)} > x_1, \ldots, X_{n-k}^{(k)} > x_k | X_{k:n} = t) dF_{k:n}(t) \\
&= \int_0^\infty P(Y_1^{(k)} > x_1 + t, \ldots, Y_{n-k}^{(k)} > x_{n-k} + t | X_{k:n} = t) dF_{k:n}(t) \\
&= \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{\bar{F}(x_j + t)}{\bar{F}(t)} \right] dF_{k:n}(t). \qquad (3.1)
\end{aligned}$$

Here and henceforth, the subscript n on a distribution, density or survival function of residual lifetimes denotes the original sample size while the superscript denotes the number of failures that have occurred. From (3.1) the joint density of the residual life lengths can be obtained by justifiably differentiating under the integral sign, to get

$$\begin{aligned}
f_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) &= \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{f(x_j + t)}{\bar{F}(t)} \right] dF_{k:n}(t) \\
&= k \binom{n}{k} \int_0^\infty \prod_{j=1}^{n-k} f(x_j + t) F^{k-1}(t) dF(t). \qquad (3.2)
\end{aligned}$$

It is obvious from (3.1) or (3.2) that $X_i^{(k)}$'s are exchangeable (it was already obvious since they were conditionally independent given $X_{k:n}$). The common marginal distribution function of the $X_i^{(k)}$'s is

$$F_n^{(k)}(x) = P(X_i^{(k)} \leq x) = k\binom{n}{k}\int_0^\infty [\bar{F}(t)]^{n-k-1}[F(t+x) - F(t)]F^{k-1}(t)dF(t). \tag{3.3}$$

The marginal density of the $X_i^{(k)}$'s can be expressed as

$$f_n^{(k)}(x) = \int_0^\infty \frac{f(t+x)}{\bar{F}(t)} f_{k:n}(t)dt, \tag{3.4}$$

where $f_{k:n}$ denotes the density of the k'th order statistic $X_{k:n}$.

Example 3.1. Suppose that $F(x) = U_{0,1}(x) = x$; $0 \leq x \leq 1$, a uniform distribution on the interval $(0, 1)$. Referring to (3.2) we find the corresponding joint density of the residual lifetimes following the k'th failure to be

$$\begin{aligned}
f_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) &= k\binom{n}{k}\int_0^1 \left[\prod_{j=1}^{n-k} I(0 < x_j + t < 1)\right] t^{k-1}dt \\
&= k\binom{n}{k}\int_0^1 I(0 < t < 1 - \max(x_i))t^{k-1}dt \\
&= \binom{n}{k}[1 - \max(x_i)]^k I(0 < x_i < 1, i = 1, 2, \ldots, n-k)
\end{aligned}$$

with corresponding marginal density (from (3.4))

$$\begin{aligned}
f_n^{(k)}(x) &= \int_0^1 \frac{I(0 < t + x < 1)}{1-t} k\binom{n}{k} t^{k-1}(1-t)^{n-k}dt \\
&= k\binom{n}{k}\int_0^{1-x} t^{k-1}(1-t)^{n-k-1}dt \\
&= k\binom{n}{k} I_{1-x}(k, n-k) \\
&= \frac{n}{n-k} I_{1-x}(k, n-k),
\end{aligned}$$

where $I_\gamma(\alpha, \beta)$ denotes the incomplete gamma function, in this case a polynomial of degree $n - k$.

Example 3.2. Suppose that $F(x) = 1 - e^{-\lambda x}, x > 0$, an exponential distribution with intensity λ. The lack of memory of the exponential distribution could be used to argue that the residual lifetimes following the k'th failure should have the same distribution as the original lifetimes. We can confirm this readily by substituting the exponential survival function in (3.1) to obtain

$$\begin{aligned}\bar{F}_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) &= \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{e^{-\lambda(x_j+t)}}{e^{-\lambda t}}\right] dF_{k:n}(t) \\ &= \prod_{j=1}^{n-k} e^{-\lambda x_j} \int_0^\infty dF_{k:n}(t) \\ &= \prod_{j=1}^{n-k} e^{-\lambda x_j}, x_j > 0, j = 1, 2, \ldots, n-k.\end{aligned}$$

This joint density of residual lives has two remarkable features. First the residual lifetimes are independent. Second the residual life distribution of a component is the same as the original life distribution of a component.

3.2 Remarks on Wearouts

Typically components degrade under usage.

Definition. F is said to be new better than used (NBU) if for every $t, x \geq 0$ we have $\bar{F}(x+t) \leq \bar{F}(x)\bar{F}(t)$. If for every $t, x \geq 0$, we have $\bar{F}(x+t) \geq \bar{F}(x)\bar{F}(t)$ then F is said to be new worse than used (NWU) (Barlow and Proschan (1975)).

We use the symbol $\leq_{st}$ to denote stochastic ordering, thus we write $X \leq_{st} Y$ (X is stochastically smaller than Y) if $P(X > x) \leq P(Y > x), \forall x \in \mathbb{R}$.

Common sense tells us that if components wear out (i.e. if F is NBU) then the residual lifetimes will be stochastically smaller than the original lifetimes. We may confirm this as follows.

Proposition 3.1 *If F is NBU(NWU), then $X_1^{(k)} \leq_{st} X_1$ ($X_1^{(k)} \geq_{st} X_1$).*

Proof. Assume that F is NBU. We can write the joint distribution function of the survival times as follows

$$F_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) = \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{F(x_j + t) - F(t)}{1 - F(t)} \right] dF_{k:n}(t).$$

The marginal distribution function of X_1 is obtained by taking the limit as $x_i \to \infty, i = 2, \ldots, n-k$. Thus

$$\begin{aligned} F_n^{(k)}(x_1) &= \int_0^\infty \frac{F(x_1 + t) - F(t)}{1 - F(t)} dF_{k:n}(t) \\ &= \int_0^\infty \frac{\bar{F}(t) - \bar{F}(x_1 + t)}{\bar{F}(t)} dF_{k:n}(t). \end{aligned}$$

Since F is NBU, we have $\bar{F}(x_1 + t) \leq \bar{F}(x_1)\bar{F}(t)$ and so

$$\begin{aligned} F_n^{(k)}(x_1) &\geq \int_0^\infty \frac{\bar{F}(t) - \bar{F}(x_1)\bar{F}(t)}{\bar{F}(t)} dF_{k:n}(t) \\ &= [1 - \bar{F}(x_1)] \int_0^\infty dF_{k:n}(t) \\ &= F(x_1). \end{aligned}$$

□

Of course if F is both NBU and NWU (i.e. if F is an exponential distribution function), then $X_1^{(k)} \stackrel{d}{=} X_1$. We now turn to characterizations related to this observation.

3.3 Characterizations

If the original component lifetime distribution F was an exponential distribution, then we have seen that the residual lifetimes following the k'th failure will be independent and will have the same marginal distribution as that of the original lifetimes. It is thus reasonable to ask whether the conditions

$$\text{(A)} \quad X_1^{(k)} \stackrel{d}{=} X_1$$

and

$$\text{(B)} \quad X_1^{(k)} \text{ and } X_2^{(k)} \text{ are independent}$$

are together or separately sufficient to guarantee that the original component lifetime distribution must be exponential. Condition (A) is readily dealt with.

Theorem 3.2 *If* $X_1^{(k)} \stackrel{d}{=} X_1$*, then* $X_1 \sim$ *exponential* (λ) *for some* $\lambda > 0$.

Proof. If $X_1^{(k)} \stackrel{d}{=} X_1$, then for every $x > 0$,

$$\bar{F}(x) = P(X_1 > x) = P(X_1^{(k)} > x) = \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)} dF_{k:n}(t).$$

Thus

$$\int_0^\infty \frac{\bar{F}(x+t) - \bar{F}(x)\bar{F}(t)}{\bar{F}(t)} dF_{k:n}(t) = 0 \quad \forall x > 0.$$

But this is an integrated Cauchy functional equation (see e.g.Rao and Shanbhag (1994)) and the only solution is if the form $\bar{F}(x) = e^{-\lambda x}, x > 0$ for some $\lambda > 0$. We are able to characterize the exponential distribution using condition (B) by imposing a rather strong regularity condition. Whether this regularity condition can be dispensed with remains an open problem. □

Theorem 3.3 *If* $X_1^{(k)}$ *and* $X_2^{(k)}$ *are independent and if*

i. $\bar{F}(x)$ *is strictly decreasing on* $(0, \infty)$ *and*

ii. for each $x > 0$, $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ *is a monotone function of* t,

then $X_1 \sim$ *exponential* (λ) *for some* $\lambda > 0$.

[Note that a sufficient condition for monotonicity of $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ for every x is that F have a monotone failure rate, i.e. is either IFR or DFR.]

Proof. For any $x_1, x_2 > 0$ we have

$$\bar{F}_{(k)}(x_1, x_2) = \int_0^\infty \left[\prod_{j=1}^{2} \frac{\bar{F}(x_j + t)}{\bar{F}(t)} \right] dF_{k:n}(t)$$

and

$$\bar{F}_{(k)}(x_1) = \int_0^\infty \left(\frac{\bar{F}(x_1 + t)}{\bar{F}(t)} \right) dF_{k:n}(t).$$

Thus if $X_1^{(k)}$ and $X_2^{(k)}$ are independent we have

$$\int_0^\infty \left[\prod_{j=1}^{2} \frac{F(x_j+t)}{\bar{F}(t)}\right] dF_{k:n}(t) = \int_0^\infty \frac{\bar{F}(x_1+t)}{\bar{F}(t)} dF_{k:n}(t) \int_0^\infty \frac{\bar{F}(x_2+t)}{\bar{F}(t)} dF_{k:n}(t).$$

We can write this as

$$cov\left(\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}, \frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}\right) = 0. \tag{3.5}$$

Recall what is sometimes called Tchebychev's second inequality. It states that for any random variable X and any two non-decreasing functions ϕ_1 and ϕ_2, then provided appropriate expectations exist we have $cov(\phi_1(X), \phi_2(X)) \geq 0$ with equality if and only if at least one of the random variables $\phi_1(X)$ and $\phi_2(X)$ is degenerate. The same conclusion holds if both ϕ_1 and ϕ_2 are non-increasing.
The assumed monotonicity of $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ for each x, together with equation (3.5) and Tchebychev's second inequality permits us to conclude that for any pair x_1, x_2, at least one of the random variables $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$ and $\frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}$ is degenerate. If for every pair x_1, x_2, both of the random variables $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}, \frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}$ are degenerate, then it follows that for every x, $\frac{\bar{F}(x+X_{k:n})}{\bar{F}(X_{k:n})}$ is degenerate, say equal to $c(x)$.

If there exists a pair x_1, x_2 for which one of the random variables $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$, $\frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}$ is not degenerate, then without loss of generality we can assume that $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$ is not degenerate, but then for every $x \neq x_1$, we must have $\frac{\bar{F}(x+X_{k:n})}{\bar{F}(X_{k:n})}$ degenerate and equal to $c(x)$, say. Thus for any $y > 0$ and any $x > 0, x \neq x_1$, we have, since $\bar{F}(x)$ is assumed to be decreasing,

$$\frac{\bar{F}(x+y)}{\bar{F}(y)} = c(x). \tag{3.6}$$

Using the right continuity of $\bar{F}$ we can define $c(x_1) = \lim_{x \downarrow x_1} c(x)$ and conclude that (3.6) holds for every $x, y > 0$. But this is Pexider's equation and thus $\bar{F}(x) = k_1 e^{-\lambda x}$ and $c(x) = k_2 e^{-\lambda x}$. Finally by considering limits as $x \to 0$, we conclude that $k_1 = k_2 = 1$. □

3.4 An extension to exchangeability

Instead of assuming that the X_i's are i.i.d we may wish to entertain the possibility that they are exchangeable. Indeed some Bayesians might argue that this is almost always more appropriate than an i.i.d assumption. Provided we interpret the concept of exchangeability in the strict deFinetti sense, then we are really dealing with conditionally independent variables. Thus we assume the existence of a random variable Z with distribution function $G(z)$, such that given $Z = z$ the X_i's are conditionally i.i.d within common marginal conditional distribution denoted by $F_z(x)$. Thus the joint distribution of $X_1, X_2, \ldots, X_n$ assumes the form:

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} F_z(x_j) \right] dG(z). \tag{3.7}$$

It then follows that the joint survival function of the residual lives of the remaining components after k failures will be of the form

$$\begin{aligned} &\bar{F}_{X_1^{(k)},X_2^{(k)},\ldots,X_{n-k}^{(k)}}(x_1, x_2, \ldots, x_{n-k}) \\ &= \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left[\prod_{j=1}^{n-k} \frac{\bar{F}_z(x_j + t)}{\bar{F}_z(t)} \right] dF_{z;k:n}(t) \right) dG(z). \end{aligned} \tag{3.8}$$

For most choices of conditional distributions F_z, this expression will be difficult to evaluate. In certain favorable cases analytic results are obtainable.

Example 3.3. Suppose that, given $Z = z$, the X_i's are conditionally independent exponential (δz) random variables. In addition suppose that $Z \sim \Gamma(\alpha, \lambda)$, i.e. that

$$f_Z(z) = \frac{\lambda^\alpha z^{\alpha-1} e^{-\lambda z}}{\Gamma(\alpha)} I(z > 0).$$

In this case we will have

$$\begin{aligned} \bar{F}^{(k)}_{X_1^{(k)},X_2^{(k)},\ldots,X_{n-k}^{(k)}|Z}(x_1, x_2, \ldots, x_{n-k}|z) &= \prod_{j=1}^{n-k} e^{-\delta z x_i} \\ &= \exp(-\delta z \sum_{j=1}^{n-k} x_j). \end{aligned}$$

Consequently the joint density of the residual lives after k failures will be given by

$$
\begin{aligned}
&F^{(k)}_{X_1^{(k)},X_2^{(k)},\ldots,X_{n-k}^{(k)}}(x_1,x_2,\ldots,x_{n-k}) \\
&= \int_0^\infty \exp(-\delta z \sum_{j=1}^{n-k} x_j) \frac{\lambda^k z^{\alpha-1} e^{-\lambda z}}{\Gamma(\alpha)} dz \\
&= (1+\frac{\delta}{\lambda}\sum_{j=1}^{n-k} x_j)^{-\alpha}, \qquad (3.9)
\end{aligned}
$$

which is a multivariate Pareto distribution. In (3.9), the $X_i^{(k)}$'s are identically distributed but only conditionally independent.

3.5 A link with mean residual life functions

For a component X_i with lifetime distribution F, the corresponding mean residual life (MRL) function ψ_F is defined as follows

$$\psi_F(t) = E(X-t|X>t) = \frac{1}{\bar{F}(t)}\int_0^\infty x f(t+x)dx. \qquad (3.10)$$

The MRL function is of much utility in actuarial, survival and reliability settings. For detailed discussion of the MLR function see Meilijson (1972), Hall and Wellner (1981), Oakes and Dasu (1990). The MRL function is related to other well known functions such as the Lorenz curve and the hazard function (cf., Arnold (1983)). Recently papers have appeared investigating the mean residual life functions of k-out-of-n systems. See for example, Bairamov et al. (2002), Asadi and Bairamov (2005), Asadi and Bairamov (2006), Li and Zhao (2006).

In fact the expected value of a residual lifetime after k failures ($X_1^{(k)}$) is directly related to the MRL function of the component lifetime distribution F, i.e. to ψ_F. We have

Theorem 3.4 $E(X_1^{(k)}) = E(\psi_F(X_{k:n})), k = 1, 2, \ldots, n-1.$

Proof. From (3.1), the density of $X_1^{(k)}$ is given by

$$f_n^{(k)}(x) = \int_0^\infty \frac{f(t+x)}{\bar{F}(t)} f_{k:n}(t)dt,$$

where $f_{k:n}(t)$ denotes the density of $X_{k:n}$. Consequently

$$\begin{aligned} E(X_1^{(k)}) &= \int_0^\infty x f_n^{(k)}(x)dx \\ &= \int_0^\infty \int_0^\infty x \frac{f(t+x)}{\bar{F}(t)} f_{k:n}(t)dtdx \\ &= \int_0^\infty \psi_F(t) f_{k:n}(t)dt = E(\psi_F(X_{k:n})). \end{aligned}$$

□

3.6 On reuse of unfailed components

Suppose that we have on hand $n-k$ unfailed units from an $n-k+1$ out of n system and that we use them to construct an $n-k-k'+1$ out of $n-k$ system. What can we say about the residual lifetimes of the $n-k-k'$ unfailed units from this $n-k-k'+1$ out of $n-k$ system. The joint distribution of the component lifetimes of the $n-k$ units used to build the second system will be only conditionally independent given $X_{k:n}$ (the failure time of the original $n-k+1$ out of n system). Their joint distribution will be given by (3.1), i.e.

$$\bar{F}_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) = \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{\bar{F}(x_j+t)}{\bar{F}(t)} \right] dF_{k:n}(t).$$

Thus for the second system, built with these used components, the joint distribution of the component lifetimes is a mixture as in (3.7) with mixing distribution $G(z) = F_{k:n}(z)$ and conditional survival functions $\bar{F}_z(x_j) = \frac{\bar{F}(x_j+z)}{\bar{F}(z)}$. We may then, using (3.8), obtain the joint survival function of the residual lifetimes of the unfailed items from the second system, i.e. the $n-k-k'+1$ out of $n-k$ system. Thus we obtain

$$\bar{F}_{X_1^{(n-k)} X_2^{(n-k)}, \ldots, X_{n-k-k'}^{(n-k)}}(x_1, x_2, \ldots, x_{n-k-k'})$$

$$= \int_0^\infty \int_0^\infty \prod_{j=1}^{n-k-k'} \frac{\bar{F}(x_j + t + z)}{\bar{F}(t+z)} dF_{z;k':n-k}(t) dF_{k:n}, \tag{3.11}$$

where $F_{z_j k':n-k}(t)$ denotes the distribution of the k^{th} order statistic from a sample of size $n-k$ from the distribution with survival function $\bar{F}(x+z)/\bar{F}(z)$. Eventually this simplifies to yield

$$\bar{F}_{X_1^{(n-k)} X_2^{(n-k)},\ldots,X_{n-k-k'}^{(n-k)}}(x_1, x_2, \ldots, x_{n-k-k'})$$

$$= \int_0^\infty \prod_{j=1}^{n-k-k'} \frac{\bar{F}(x_j + u)}{\bar{F}(u)} dF_{k+k':n}(u)$$

confirming the retrospectively obvious result that the residual lives of the remaining components after serving in both systems, correspond in distribution to the residual lives of the remaining components when a $n-k-k'+1$ out of n system has failed. We are indeed waiting first for k failures and then k' more failures among the original n components.

If only $n-p$ of the surviving components from the $n-k+1$ out of n system are used to construct an $n-p-k'+1$ out of $n-p$ system, equation (3.11) must be slightly modified to describe the residual lives of the surviving components in the second system. We will have

$$\bar{F}_{X_1^{(n-p)},X_2^{(n-p)},\ldots,X_{n-p-k'}^{(n-p)}}(x_1, x_2, \ldots, x_{n-p-k'})$$

$$= \int_0^\infty \int_0^\infty \prod_{j=1}^{n-p-k'} \frac{\bar{F}(x_j + t + z)}{\bar{F}(t+z)} dF_{z_j k'+n-p}(t) dF_{k,n}(z). \tag{3.12}$$

Only in very special cases will it be possible to simplify this expression. For example, if the original components had exponential (λ) lifetime distributions then the lack of memory property guarantees that the residual lifetimes of the surviving components in the second system (the $n-p-k'+1$ out of $n-p$ system) will again have independent exponential (λ) distributions. Substitution of $\bar{F}(x) = e^{-\lambda x}$ in (3.12) will confirm this conclusion.

Chapter 4

Residual Lifetimes of Remaining Progressively Type-I Censored Order Statistics

Suppose n independent units are placed on a life-test with the corresponding failure times $X_1, X_2, ..., X_n$ being identically distributed with c.d.f $F(x)$ and p.d.f $f(x)$. Suppose that in a given time t the number of failed components are known and we are interested in residual lifetimes of the remaining components that survives after time t. Let $\xi(t)$ denotes the number of failed components up to time t. Given $\xi(t) = k \Leftrightarrow (X_{k:n} < t < X_{k+1:n})$ the residual lifetimes of the remaining can be represented as

$$F^{(t,k)}_{n-k}(x_1, x_2, ...x_{n-k}) = P\{X_{k+1:n} - t \le x_1, X_{k+2:n} - t \le x_2, ...X_{n:n} - t \le x_{n-k} | \, \xi(t) = k\}$$

$$F^{(t,k)}_{n-k}(x_1, x_2, ...x_{n-k}) = \frac{P\{X_{k:n} < t, \ t < X_{k+1:n} - t \le x_1, ..., X_{n:n} - t \le x_{n-k}\}}{P\{X_{k:n} < t < X_{k+1:n}\}}$$

$$= \frac{\begin{array}{c} P\{X_{k:n} < t, \ t < X_{k+1:n} \le x_1 + t, X_{k+1:n} < X_{k+2:n} \le x_2 + t, \\ ..., X_{n-1:n} < X_{n:n} \le x_{n-k} + t\} \end{array}}{P\{X_{k:n} < t < X_{k+1:n}\}}$$

$$= \frac{\binom{n}{k} F(t)^k \int\limits_{t}^{x_1+t} \int\limits_{u_1}^{x_2+t} \int\limits_{u_2}^{x_3+t} ... \int\limits_{u_{n-k-1}}^{x_{n-k}+t} dF(u_{n-k}) dF(u_{n-k-1}) ... dF(u_1)}{\binom{n}{k} F(t)^k (1 - F(t))^{n-k}}$$

$$= \frac{\int\limits_{t}^{x_1+t} \int\limits_{u_1}^{x_2+t} \int\limits_{u_2}^{x_3+t} ... \int\limits_{u_{n-k-1}}^{x_{n-k}+t} dF(u_{n-k}) dF(u_{n-k-1}) ... dF(u_1)}{(1 - F(t))^{n-k}}.$$

4.1 The joint and the marginal distribution of the residual lifelengths of remaining components

Theorem 4.1 *Let $X_1, X_2, ..., X_n$ denote the corresponding failure times of n components being identically distributed with c.d.f $F(x)$ and p.d.f $f(x)$. Given $\xi(t) = k$ the joint distribution of the residual lifetimes of the remaining $n-k$ items for $n - k \ge 2$ can be found by the recursive formula given below*

$$F^{(t,k)}_{n-k}(x_1, x_2, ...x_{n-k}) =$$

$$\frac{\begin{array}{l} (1 - F(t))^{n-k-1} \Big[F(x_{n-k} + t) F^{(t,k)}_{n-k-1}(x_1, x_2, ...x_{n-k-1}) \\ \quad - \sum\limits_{j=2}^{n-k-1} \frac{F(x_j + t)}{(n-k-j+1)} A^{(t,k)}_{\max n-k-1, j}(x_1, x_2, ...x_{n-k-1}) \Big] + (-1)^{n-k+1} \frac{F^{n-k}(x_1+t) - F^{n-k}(t)}{(n-k)!} \end{array}}{(1 - F(t))^{n-k}}$$

where $A^{(t,k)}_{\max n-k-1, j}(x_1, x_2, ..., x_{n-k-1})$ denotes the terms of the joint distribution function $F^{(t,k)}_{n-k-1}(x_1, x_2, ...x_{n-k-1})$, whose maximum index is j, and $F^{(t)}_1(x_1) = \frac{F(x_1+t) - F(t)}{1 - F(t)}$

Using theorem 4.1 one can evaluate the joint distributions of the residual lifelenghts for $n-k=2,3$ respectively.

$$F_2^{(t,k)}(x_1,x_2) = \frac{F(x_2+t)\left[F(x_1+t)-F(t)\right] - \frac{F^2(x_1+t)-F^2(t)}{2!}}{(1-F(t))^2}$$

$$A_{\max 2,2}^{(t,k)}(x_1,x_2) = \frac{F(x_2+t)\left[F(x_1+t)-F(t)\right]}{(1-F(t))^2}$$

$$F_3^{(t,k)}(x_1,x_2,x_3) = \frac{(1-F(t))^2\left[F(x_3+t)F_2^{(t,k)}(x_1,x_2) - \frac{F(x_2+t)}{2}A_{\max 2,2}^{(t,k)}(x_1,x_2)\right] + \frac{F^3(x_1+t)-F^3(t)}{3!}}{(1-F(t))^3}$$

$$= \frac{\begin{array}{c} F(x_3+t)F(x_2+t)\left[F(x_1+t)-F(t)\right] - F(x_3+t)\frac{F^2(x_1+t)-F^2(t)}{2!} \\ -\frac{F^2(x_2+t)}{2}\left[F(x_1+t)-F(t)\right] + \frac{F^3(x_1+t)-F^3(t)}{3!}\end{array}}{(1-F(t))^3}$$

$$A_{\max 3,2}^{(t,k)}(x_1,x_2,x_3) = \frac{-\frac{F^2(x_2+t)}{2}\left[F(x_1+t)-F(t)\right]}{(1-F(t))^3}$$

$$A_{\max 3,3}^{(t,k)}(x_1,x_2,x_3) = \frac{F(x_3+t)F(x_2+t)\left[F(x_1+t)-F(t)\right] - F(x_3+t)\frac{F^2(x_1+t)-F^2(t)}{2!}}{(1-F(t))^3}$$

The marginal survival function of the residual lifetimes $X_{k+r:n}$ $r=1,2...n-k$ given $(X_{k:n} < t < X_{k+1:n})$ can be expressed as

$$= P\{X_{k+r:n} - t > x_r \,|X_{k:n} < t < X_{k+1:n}\}$$
$$= \frac{P\{X_{k+r:n} > x_r + t, X_{k:n} < t < X_{k+1:n}\}}{P\{X_{k:n} < t < X_{k+1:n}\}}$$

$$P\{X_{k+r:n} > x_r + t, X_{k:n} < t < X_{k+1:n}\} =$$
$$\binom{n}{k}(F(t))^k \sum_{j=n-k-r+1}^{n-k} \binom{n-k}{j} (\overline{F}(x_r+t))^j (F(x_r+t)-F(t))^{n-k-j}.$$

Since

$$P\{X_{k:n} < t < X_{k+1:n}\} = \binom{n}{k}(F(t))^k(\overline{F}(t))^{n-k}$$

one can find the marginal survival functions of the residual lifetimes for $r = 1,2...n-k$ as

$$P\{X_{k+r:n} - t > x_r \,|X_{k:n} < t < X_{k+1:n}\} = \frac{P\{X_{k+r:n} > x_r + t, X_{k:n} < t < X_{k+1:n}\}}{P\{X_{k:n} < t < X_{k+1:n}\}}$$
$$= \sum_{j=n-k-r+1}^{n-k} \binom{n-k}{j} \left(\frac{\overline{F}(x_r+t)}{\overline{F}(t)}\right)^j \left(1 - \frac{\overline{F}(x_r+t)}{\overline{F}(t)}\right)^{n-k-j}.$$

Chapter 5

Residual Lifetimes of Remaining Progressively Type-II Right Censored Order Statistics

Normally in progressively type-II right censoring n units are placed on test at time zero. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then immediately following the second observed failure, R_2 surviving units are removed from the test at random. This process continues at the time of the m-th observed failure remaining $R_m = n - R_1 - R_2 - ... - R_{m-1} - m$ units are all removed from the experiment. Now suppose after m-th observed failure we do not remove all remaining units. We are interested on the remaining lifetimes of the $n - R_1 - R_2 - ... - R_{m-1} - m = R_m$ survived units given the time of the m-th failure.

For any $m \in \{1, 2...n\}$ we will use the notation $X_1^{(m)}, X_2^{(m)}, ..., X_{R_m}^{(m)}$ to denote the residual lifetimes of the R_m units still functioning at the time of mth failure. Denote $Y_i^{(m)}$, $i = 1, 2...R_m$. randomly ordered values of $X_{i:m:R_m}$, $i = 1, 2...R_m$. Given $X_{m:m:n} = x$ these $Y_i^{(m)}$ will be i.i.d with common survival function $\overline{F}(x + y)/\overline{F}(x)$. Then residual lifetimes of the remaining components after m failures can be represented as

$$X_i^{(m)} = Y_i^{(m)} - X_{m:m:n},\ i = 1, 2...R_m.$$

Using $F_{X_{m:m:n}}$ to denote the distribution of $X_{m:m:n}$ we can obtain the joint survival function of the residual lifelenghts as follows

$$\begin{aligned}
&\overline{F}_n^{(m)}(x_1, x_2, ...x_{n-R_1-R_2-...-R_{m-1}-m}) \\
&= P\{X_1^{(m)} > x_1,\ X_2^{(m)} > x_2, ...,\ X_{R_m}^{(m)} > x_{R_m}\} \\
&= \int_0^\infty P\{X_1^{(m)} > x_1, ...,\ X_{R_m}^{(m)} > x_{R_m} \,|X_{m:m:n} = t\} dF_{X_{m:m:n}}(t) \\
&= \int_0^\infty P\{Y_1^{(m)} > x_1 + t, ...,\ Y_{R_m}^{(m)} > x_{R_m} + t \,|X_{m:m:n} = t\} dF_{X_{m:m:n}}(t) \\
&= \int_0^\infty \left[\prod_{j=1}^{R_m} \frac{\overline{F}(x_j + t)}{\overline{F}(t)} \right] dF_{X_{m:m:n}}(t). \qquad (5.1)
\end{aligned}$$

The joint density of residual lifelenghts can be obtained by differentiating under the integral sign to get

$$\int_0^\infty \left[\prod_{j=1}^{R_m} \frac{f(x_j + t)}{\overline{F}(t)} \right] dF_{X_{m:m:n}}(t). \qquad (5.2)$$

The marginal density of $X_i^{(m)}$ can be expressed as

$$f_n^{(m)}(x) = \int_0^\infty \frac{f(x + t)}{\overline{F}(t)} f_{X_{m:m:n}}(t) dt. \qquad (5.3)$$

5.1 Aging of the remaining components

Proposition 5.1 *If F is NBU (NWU), then $X_1^{(m)} \leq_{st} X_1$ ($X_1^{(m)} \geq_{st} X_1$).*

Proof. Assume that F is NBU. We can write the joint distribution function of the survival times as follows

$$F_n^{(m)}(x_1, x_2, \ldots, x_{R_m}) = \int_0^\infty \left[\prod_{j=1}^{R_m} \frac{F(x_j + t) - F(t)}{1 - F(t)} \right] dF_{X_{m:m:n}}(t).$$

The marginal distribution function of X_1 is obtained by taking the limit as $x_i \to \infty, i = 2, \ldots, n-k$. Thus

$$\begin{aligned} F_n^{(m)}(x_1) &= \int_0^\infty \frac{F(x_1 + t) - F(t)}{1 - F(t)} dF_{X_{m:m:n}}(t) \\ &= \int_0^\infty \frac{\bar{F}(t) - \bar{F}(x_1 + t)}{\bar{F}(t)} dF_{X_{m:m:n}}(t). \end{aligned}$$

Since F is NBU, we have $\bar{F}(x_1 + t) \le \bar{F}(x_1)\bar{F}(t)$ and so

$$\begin{aligned} F_n^{(m)}(x_1) &\ge \int_0^\infty \frac{\bar{F}(t) - \bar{F}(x_1)\bar{F}(t)}{\bar{F}(t)} dF_{X_{m:m:n}}(t) \\ &= [1 - \bar{F}(x_1)] \int_0^\infty dF_{X_{m:m:n}}(t) \\ &= F(x_1). \end{aligned}$$

□

5.2 Characterizations

Theorem 5.2 *If* $X_1^{(m)} \stackrel{d}{=} X_1$, *then* $F \equiv Exp(\lambda)$, $\lambda > 0$.

Proof. If $X_1^{(m)} \stackrel{d}{=} X_1$ then for every $x > 0$,

$$\bar{F}(x) = P(X_1 > x) = P(X_1^{(m)} > x) = \int_0^\infty \frac{\bar{F}(x + t)}{\bar{F}(t)} dF_{X_{m:m:n}}(t).$$

Thus

$$\int_0^\infty \frac{\bar{F}(x + t) - \bar{F}(x)\bar{F}(t)}{\bar{F}(t)} dF_{X_{m:m:n}}(t) = 0 \quad \forall x > 0.$$

But this is an integrated Cauchy functional equation (see e.g.Rao and Shanbhag (1994)) and the only solution is if the form $\bar{F}(x) = e^{-\lambda x}, x > 0$ for some $\lambda > 0$. □

5.3 An extension to exchangeability

Instead of assuming that the X_i's are i.i.d we may wish to entertain the possibility that they are exchangeable. It follows from (3.7) that the joint survival function of the residual lives of the remaining components after m failures will be of the form

$$\begin{aligned}
&\bar{F}_{X_1^{(m)},X_2^{(m)},\ldots,X_{n-R_m}^{(m)}}(x_1, x_2, \ldots, x_{R_m}) \\
&= \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left[\prod_{j=1}^{R_m} \frac{\bar{F}_z(x_j + t)}{\bar{F}_z(t)} \right] dF_{z;m:m:n}(t) \right) dG(z). \qquad (5.4)
\end{aligned}$$

5.4 A link with mean residual life functions

The expected value of a residual lifetime after m failures ($X_1^{(m)}$) is directly related to the MRL function of the component lifetime distribution F, i.e. to ψ_F.

Theorem 5.3 $E(X_1^{(m)}) = E(\psi_F(X_{m:m:n})), m = 1, 2, \ldots, n-1.$

Proof. The density of $X_1^{(m)}$ is given by

$$f_n^{(m)}(x) = \int_0^{\infty} \frac{f(x+t)}{\overline{F}(t)} f_{X_{m:m:n}}(t)dt,$$

where $f_{X_{m:m:n}}(t)$ denotes the density of $X_{m:m:n}$. Consequently

$$\begin{aligned}
E(X_1^{(m)}) &= \int_0^{\infty} x f_n^{(m)}(x)dx \\
&= \int_0^{\infty} \int_0^{\infty} x \frac{f(t+x)}{\bar{F}(t)} f_{X_{m:m:n}}(t)dtdx \\
&= \int_0^{\infty} \psi_F(t) f_{m:m:n}(t)dt = E(\psi_F(X_{m:m:n})).
\end{aligned}$$

□

Chapter 6

Residual Lifetimes of the Remaining Sequentially Ordered Statistics

When modelling of failure times for components of the system having i.i.d components it is assumed that the failure of one component does not affect the functioning ones. If that is not the case for instance if the failure of a component puts added stress or load on the remaining components the model involving sequentially ordered statistics can be appropriately used in the analysis of the system. Gurler(2010) extended some results on the joint distribution of the residual lifetimes of the remaining components in an ordinary $(n-k+1)$-out-of-n system presented in Bairamov and Arnold to the case of the sequential $(n-k+1)$-out-of-n system.

Let $X_1, X_2, \ldots, X_n$ be the lifetimes of the components each following continuos distribution functions $F_1, F_2, \ldots, F_n$ and respective density functions $f_1, f_2, \ldots, f_n$. To simplify the presentation, we assume $F_i^{-1}(0+) = 0$ and $F_i^{-1}(1) = \infty$, $1 \leq i \leq s$. Moreover, let $X_*^{(1)}, X_*^{(2)}, \ldots, X_*^{(n)}$ denote sequential order statistics based on $F_1, F_2, \ldots, F_n$ that describe the failure times in a sequential $(n-k+1)$-out-of-n system. Then, the lifetime of the sequential $(n-k+1)$-out-of-n system is modeled by the kth sequential order statistic $X_*^{(k)}$, $1 \leq k \leq n$.

Let $X_j^{(k+1)}, j = 1, 2, \ldots, n-k$, represent the overall lifetimes of the remaining components in the sequential $(n-k+1)$-out-of-n system after the kth failure,i.e. after the system itself has failed. Then, the residual lifetimes of the remaining components after the occurrence of the kth failure are given by

$$Z_j^{(k+1)} = X_j^{(k+1)} - X_*^{(k)}, \quad j = 1, 2, \ldots, n-k.$$

It is known that the residual lifetimes of these components are conditionally independent given the k^{th} failure time $X_*^{(k)}$ in the system. If $F^{X_*^{(k)}}(s)$ denotes the distribution function of the kth sequential order statistic $X_*^{(k)}$, then the joint survival function of the residual lifetimes of the $n-k$ components after kth failure is given by

$$\begin{aligned}
\overline{F}^{(k+1)}(z_1, \ldots, z_{n-k}) &= P(Z_1^{(k+1)} > z_1, \ldots, Z_{n-k}^{(k+1)} > z_{n-k}) \\
&= \int_0^\infty P(Z_1^{(k+1)} > z_1, \ldots, Z_{n-k}^{(k+1)} > z_{n-k} | X_*^{(k)} = s) dF^{X_*^{(k)}}(s) \\
&= \int_0^\infty P(X_1^{(k+1)} > z_1 + s, \ldots, X_{n-k}^{(k+1)} > z_{n-k} + s | X_*^{(k)} = s) dF^{X_*^{(k)}}(s) \\
&= \int_0^\infty \prod_{i=1}^{n-k} \left(\frac{1 - F_{k+1}(z_i + s)}{1 - F_{k+1}(s)} \right) dF^{X_*^{(k)}}(s). \qquad (6.1)
\end{aligned}$$

Analogously, the joint distribution function can be derived as follows.

$$F^{(k+1)}(z_1, \ldots, z_{n-k}) = \int_0^\infty \prod_{i=1}^{n-k} \left(\frac{F_{k+1}(z_i + s) - F_{k+1}(s)}{1 - F_{k+1}(s)} \right) dF^{X_*^{(k)}}(s). \qquad (6.2)$$

A particular choice of the involved distribution functions $F_1, \ldots, F_n$ is given by

$$F_i = 1 - (1 - F)^{\alpha_i}, \quad 1 \le i \le n, \qquad (6.3)$$

where F is an absolutely continuous distribution function with density f and $\alpha_1, \ldots, \alpha_n$ are positive real numbers used to indicate the influence of a failure on the remaining components (see Kamps 1995). Utilizing this representation, the joint survival function of residual lifetimes after k failures simplifies to

$$\overline{F}^{(k+1)}(z_1, \ldots, z_{n-k}) = \int_0^\infty \prod_{i=1}^{n-k} \left(\frac{\overline{F}(z_i + s)}{\overline{F}(s)} \right)^{\alpha_{k+1}} dF^{X_*^{(k)}}(s),$$

where $\overline{F} = 1 - F$ denotes the survival function of F.
For the joint density function of the residual lifetimes, we obtain

$$f^{(k+1)}(z_1, \ldots, z_{n-k}) = \alpha_{k+1} \int_0^\infty \prod_{i=1}^{n-k} \frac{\left(f(z_i+s)\overline{F}(z_i+s)\right)^{\alpha_{k+1}-1}}{\left(\overline{F}(s)\right)^{\alpha_{k+1}}} dF^{X_*^{(k)}}(s). \quad (6.4)$$

Let $h^{(k+1)}$ and $H^{(k+1)}$ denote the common marginal density and marginal distribution functions of the $Z_i^{(k+1)}$'s. Then the common marginal distribution function of the $Z_i^{(k+1)}$'s is,

$$\begin{aligned} H^{(k+1)}(z) &= P\left(Z_i^{(k+1)} \le z\right) \\ &= \int_0^\infty P\left(Z_i^{(k+1)} \le z | X_*^{(k)} = s\right) dF^{X_*^{(k)}}(s) \\ &= \int_0^\infty P\left(X_i^{(k+1)} \le z + s | X_*^{(k)} = s\right) dF^{X_*^{(k)}}(s) \\ &= \int_0^\infty \left[\frac{F_{k+1}(z+s) - F_{k+1}(s)}{1 - F_{k+1}(s)}\right] dF^{X_*^{(k)}}(s). \end{aligned} \quad (6.5)$$

Using the relation in (6.3), marginal distribution and the marginal density of $Z_i^{(k+1)}$'s are written as

$$H^{(k+1)}(z) = \int_0^\infty \left[\frac{\left(\overline{F}(s)\right)^{\alpha_{k+1}} - \left(\overline{F}(z+s)\right)^{\alpha_{k+1}}}{\left(\overline{F}(s)\right)^{\alpha_{k+1}}}\right] dF^{X_*^{(k)}}(s), \quad (6.6)$$

$$h^{(k+1)}(z) = \int_0^\infty \left[\frac{\alpha_{k+1} f(z+s)\left(\overline{F}(z+s)\right)^{\alpha_{k+1}-1}}{\left(\overline{F}(s)\right)^{\alpha_{k+1}}}\right] dF^{X_*^{(k)}}(s). \quad (6.7)$$

6.1 Aging of the remaining components

Proposition 6.1 *If F is NBU (NWU), then $Z_1^{(k+1)} \le_{st} Z_1$ ($Z_1^{(k+1)} \ge_{st} Z_1$).*

Proof. The marginal distribution of $Z_1^{(k+1)}$ has the following form:

$$H^{(k+1)}(z_1) = \int_0^\infty \frac{\left(\overline{F}(s)\right)^{\alpha_{k+1}} - \left(\overline{F}(z_1+s)\right)^{\alpha_{k+1}}}{\left(\overline{F}(s)\right)^{\alpha_{k+1}}} dF^{X_*^{(k)}}(s).$$

If F is NBU, we have

$$\left(\overline{F}(z_1+s)\right)^{\alpha_{k+1}} \leq \left(\overline{F}(z_1)\right)^{\alpha_{k+1}} \left(\overline{F}(s)\right)^{\alpha_{k+1}}.$$

Hence,

$$\begin{aligned} H^{(k+1)}(z_1) &\geq \int_0^{\infty} \frac{\left(\overline{F}(s)\right)^{\alpha_{k+1}} - \left(\overline{F}(z_1)\right)^{\alpha_{k+1}} \left(\overline{F}(s)\right)^{\alpha_{k+1}}}{\left(\overline{F}(s)\right)^{\alpha_{k+1}}} dF^{X_*^{(k)}}(s) \\ &\geq 1 - \left(\overline{F}(z_1)\right)^{\alpha_{k+1}} \\ &= F_{k+1}(z_1). \end{aligned}$$

□

6.2 Characterizations

Theorem 6.2 *If $Z_1^{(k+1)} \stackrel{d}{=} Z_1$, then $F \equiv Exp(\lambda)$, $\lambda > 0$.*

Proof. If $Z_1^{(k+1)} \stackrel{d}{=} Z_1$, then for $z > 0$,

$$\overline{H}^{(k+1)}(z) = \int_0^{\infty} \left[\frac{\overline{F}_{k+1}(z+s)}{\overline{F}_{k+1}(s)}\right] dF^{X_*^{(k)}}(s) = \overline{F}_{k+1}(z)$$

Thus, the following assertion holds:

$$\begin{aligned} &= \int_0^{\infty} \left[\frac{\overline{F}_{k+1}(z+s)}{\overline{F}_{k+1}(s)}\right] dF^{X_*^{(k)}}(s) - \int_0^{\infty} \overline{F}_{k+1}(z) dF^{X_*^{(k)}}(s) = 0, \\ &= \int_0^{\infty} \left[\frac{\overline{F}_{k+1}(z+s) - \overline{F}_{k+1}(z)\overline{F}_{k+1}(s)}{\overline{F}_{k+1}(s)}\right] dF^{X_*^{(k)}}(s) = 0, \text{ for } z > 0. \end{aligned}$$

This satisfies the integrated Cauchy functional equation (Rao and Shanbhag 1994) and the solution is only based on the exponential distribution for some $\lambda > 0$. □

6.3 Exchangeability of the residual lifetimes

The concept of exchangeability introduced by deFinetti allows the more flexible modeling by forcing Bayesian approach. It is important to notice that when X_i's are considered to be a random sample from some model, then they are necessarily exchangeable. If we assume that there exists a random variable Y with distribution function $G(y)$, given $Y = y$, X_i's are conditionally independent with common marginal conditional distribution denoted by $F_Y(x)$. Thus the joint distribution of $X_1, X_2, \ldots, X_n$ assumes the form:

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} F_Y(x_j)\right] dG(y).$$

Similarly, the joint distribution of $Z_i^{(k+1)}$'s follows

$$\overline{F}^{(k+1)}(z_1,\ldots,z_{n-k}) = \int_{-\infty}^{\infty} \left[\int_{0}^{\infty} \prod_{i=1}^{n-k} \left(\frac{1-F_{Y,k+1}(z_i+s)}{1-F_{Y,k+1}(s)}\right) dF^{X_*^{(k)}}(s)\right] dG(y). \tag{6.8}$$

If we suppose that, given $Y = y$, $Z_i^{(k+1)}$'s are conditionally independent exponential random variables, then

$$\overline{F}^{(k+1)}(z_1,\ldots,z_{n-k}) = \int_{-\infty}^{\infty} \prod_{i=1}^{n-k} \left(\overline{F}_Y(z_i)\right)^{\alpha_{k+1}} dG(y). \tag{6.9}$$

6.4 Expected value of the residual lifetimes

The concept of mean residual life is based on conditional expectations and has been much interest in the actuarial science, survival studies and reliability theory. The mean residual life function $\Psi(s)$ of a component, with distribution function F related to a lifelength X, is defined by the following expectation of $X - s$ given $X > s$:

$$\Psi(t) = E\left(X - s | X > s\right).$$

If we denote the survival function for a component at the $(k+1)$th level as $S^{(k+1)}(x|s)$, the mean residual life function $\Psi^{(k+1)}(s)$ can be defined as follows:

$$\begin{aligned}\Psi^{(k+1)}(s) &= E\left(X_i^{(k+1)} - s|X_i^{(k+1)} > s\right)\\ &= \int_0^\infty S^{(k+1)}(x|s)dx. \end{aligned} \tag{6.10}$$

The survival function of the remaining components after the kth failure in the system has the following form:

$$\begin{aligned}S^{(k+1)}(x|s) &= P(X_i^{(k+1)} - s|X_i^{(k+1)} > s)\\ &= \frac{P(X_i^{(k+1)} > x+s)}{P(X_i^{(k+1)} > s)}\\ &= \frac{\overline{F}_{k+1}(x+s)}{\overline{F}_{k+1}(s)}.\end{aligned}$$

Utilizing (6.10) we have,

$$\Psi^{(k+1)}(s) = \int_0^\infty \frac{\overline{F}_{k+1}(x+s)}{\overline{F}_{k+1}(s)}dx, \text{ for } s \geq 0. \tag{6.11}$$

Theorem 6.3 *The expected value of a residual lifetime after k failures is directly related to MRL function of the* $(k+1)$*th level. That is*

$$E\left(Z_1^{(k+1)}\right) = E\left(\Psi^{(k+1)}(X_*^{(k)})\right).$$

Proof. The result follows directly from the expectation of $Z_i^{(k+1)}$.

$$\begin{aligned}E\left(Z_1^{(k+1)}\right) &= \int_0^\infty P\left(Z^{(k+1)} > z\right)dz\\ &= \int_0^\infty\int_0^\infty P\left(X^{(k+1)} > z+s|X_*^{(k)} = s\right)dF^{X_*^{(k)}}(s)dz\\ &= \int_0^\infty\int_0^\infty \frac{\overline{F}_{k+1}(z+s)}{\overline{F}_{k+1}(s)}dF^{X_*^{(k)}}(s)dz\\ &= \int_0^\infty \Psi^{(k+1)}(s)dF^{X_*^{(k)}}(s)\\ &= E\left(\Psi^{(k+1)}(X_*^{(k)})\right).\end{aligned}$$

□

Chapter 7

Conclusion

The subject of residual lifelengths has a lot of applications in real life. It is logical to investigate distributional properties and characterizations of the residual lifetimes of the remaining components to have foreknowledge incase they will be reused in other systems. It is seen that even under the classical assumption that the original lifetimes were i.i.d, it will turn out that the residual lifetimes of the remaining items will be exchangeable, but typically not independent. They will be conditionally independent under different censoring schemes.

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